This week

1. Section 8.1: integration by parts
2. Section 8.2: trigonometric integrals
Integrating the product rule

\[ \int f'(x)g(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx \]

Proof:

- Recall the product rule for differentiation:
  \[
  \frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).
  \]

- Integrating both sides gives
  \[
  \int \frac{d}{dx} f(x)g(x) \, dx = \int f'(x)g(x) + f(x)g'(x) \, dx
  \]

- Moving the second term to the left gives the boxed formula.

For definite integrals the rule reads as

\[
\int_{a}^{b} f'(x)g(x) \, dx = [f(x)g(x)]_{a}^{b} - \int_{a}^{b} f(x)g'(x) \, dx
\]

\[
= f(b)g(b) - f(a)g(a) - \int_{a}^{b} f(x)g'(x) \, dx.
\]
Example 1, alternative 1

\[ \int xe^x \, dx = ?? \]

Example 1: choose wisely...

\[ \int xe^x \, dx = ?? \]
Explicit renaming

\[ \int f'(x)g(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx \]

Notate \( f(x) = u \) and \( g(x) = v \), then the rule becomes

\[ \int u'v \, dx = uv - \int uv' \, dx \]

Example 1, alternative 2

\[ \int xe^x \, dx = ?? \]
Example 2

\[ \int x^2 e^{-x} \, dx = ?? \]

Example 3

\[ \int x \ln(x) \, dx = ?? \]
Example 4

\[ \int \ln(x) \, dx = ?? \]

Implicit integration by parts

- With the explicit renaming \( f(x) = u \) and \( g(x) = v \):
  \[ \int u' v \, dx = uv - \int u v' \, dx \]

- Note that \( du = u' \, dx \) and \( dv = v' \, dx \). Therefore the rule can be memorized as follows:
  \[ \int v \, du = uv - \int u \, dv \]

- You can even do this without renaming \( f \) and \( g \):
  \[ \int g(x) \, df(x) = f(x)g(x) - \int f(x) \, dg(x) \]
Abuse of the differential Recap 2.2

- If \( u = g(x) \) then
  \[
  du = g'(x) \, dx.
  \]

- Write
  \[
  d(g(x)) = g'(x) \, dx.
  \]

- From right to left: differentiate, from left to right: integrate:

  \[
  \begin{align*}
    & d(x^2 + 1) & d(\frac{1}{3}x^3) & d(e^{2x}) \\
    & 2x \, dx & x^2 \, dx & 2e^{2x} \, dx
  \end{align*}
  \]

- You may add an arbitrary constant to the right hand side:

  \[
  2x \, dx = d x^2 = d (x^2 + 36).
  \]

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Example 5 2.3

\[
\int (2x + 1)e^x \, dx = ??
\]
Example 6, first attempt

\[ \int x^3 e^{x^2} \, dx = ?? \]

Example 6, alternative 1

\[ \int x^3 e^{x^2} \, dx = ?? \]
Example 6, alternative 2

\[ \int x^3 e^{x^2} \, dx = ?? \]

Example 7

\[ \int e^{\sqrt{x}} \, dx = ?? \]
Example 8

\[ I = \int e^x \cos(x) \, dx = ?? \]

Example 9

\[ \int \cos(\ln x) \, dx = ?? \]
Powers of sines and cosines

4.1

Let $m$ and $n$ be non-negative integers.

$$\int \sin^m x \cos^n x \, dx = ??$$

The following formulas are useful:

- $\sin^2 x + \cos^2 x = 1$
- $\sin x \cos x = \frac{1}{2} \sin(2x)$
- $\sin^2 x = \frac{1}{2} (1 - \cos(2x))$
- $\cos^2 x = \frac{1}{2} (1 + \cos(2x))$

Example 10, alternative 1

$$\int \sin^2 x \cos^2 x \, dx = ??$$
Example 10, alternative 2

\[ \int \sin^2 x \cos^2 x \, dx = ?? \]

Example 11

\[ \int \cos^4 x \, dx = ?? \]
Example 12

\[ \int \cos x \sin^2 x \, dx = ?? \]

Trigonometry: example 14

\[ \int \cos x \sin(2x) \, dx = ?? \]